Metric Spaces and Topology
Lecture 14

Obs. Meagre sets form a σ-ideal, i.e. the collection of such sets is closed downward under $\subseteq$ and ctbl unions.

The complement of a meagre set is called *comeagre*.

Upgrade property. Let X be a topological space and $S \subseteq X$.

(a) S is meager $\iff$ $S \subseteq$ ctbl union of closed nd. sets.
(b) S is comeagre $\iff$ $S \supseteq$ ctbl intersection of dense open sets.
In particular, dense G_δ sets are comeagre.
(The converse holds in Baire spaces.)

Def. A metric space (top. space) is called *0-dimensional* if it admits a basis of clopen sets (equivalently, open sets with empty boundary). More generally, the empty space is defined to have *dimension -1*, and a space is *n-dimensional*, for $n \in \mathbb{N}$, if it admits a basis of open set whose boundary is $(n-1)$-dimensional.

Examples. ○ Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$, more generally, $A^{\mathbb{N}}$ for any set A.

○ $\mathbb{R}\setminus\mathbb{Q}$ is zero-dim. because the open intervals with rational endpoints form a basis and they are clopen in $\mathbb{R}\setminus\mathbb{Q}$. (Indeed, if $q_1 < q_2$ are rationals, then $(q_1, q_2) \cap (\mathbb{R}\setminus\mathbb{Q}) = [q_1, q_2] \cap (\mathbb{R}\setminus\mathbb{Q})$.)

○ $\mathbb{R}^n$ is (topologically) n-dimensional (not just linearly).

HW $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$. (Hint: continued fraction expansion.)

Prop. Every 2^{nd} ctbl top space X admits a zero-dim comeagre subspace.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a ctbl basis. Recall that ∂U_n is n.d., so $X \setminus \bigcup_n \partial U_n$ is comeagre and zero-dim. $\square$

Baire spaces. We saw that meagre sets form a σ-ideal, and intuitively, meagre sets should be "small."

But in some spaces, an open set or even the whole space is meagre, e.g. $X := \mathbb{Q}$.

Def. A top. space X is called **Baire** if nonempty open sets are nonmeagre. In particular, comeagre sets are nonmeagre.

Obs. An open subset of a Baire is also Baire.

Prop. For a metric space X, TFAE:

(1) X is Baire.

(2) Comeagre sets are dense.

(3) Intersection of ctbly many dense open sets is dense.

Proof. (1) $\Rightarrow$ (2). Complement of comeagre is meagre hence cannot contain a nonempty open set.

(2) $\Leftrightarrow$ (3). By the upgrade property.

(2) $\Rightarrow$ (1). If an open set U is meagre, then U^c is comeagre, hence dense so $U = \emptyset$. $\square$

Upgrade for Baire spaces. In a Baire space,

(b′) a set is comeagre $\Leftrightarrow$ it $\supseteq$ dense G_δ set.

Proof. Follows from the upgrade property (b) above and statement (3) in the equivalences above. $\square$

In Baire space, the intuition is as follows:

meagre $\longleftrightarrow$ small, null, neglegible

nonmeagre $\longleftrightarrow$ nonsmall, positive measure

comeagre $\longleftrightarrow$ almost everything, conull

But which spaces are Baire?

Baire "Category" Theorem. Complete metric spaces are Baire.

Instead of proving this directly, we will show that complete metric spaces are Choquet and Choquet spaces are Baire.

Def. In a topological space X, the Choquet game is played as follows:

Player 1. U_0 $\quad$ U_1 $\quad$ U_2 $\quad \dots$

Player 2. $\quad$ V_0 $\quad$ V_1 $\quad$ V_2

where $U_n, V_n \subseteq X$ are open sets s.t. $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$

Player 2 wins if $\bigcap_n U_n \left(= \bigcap_n V_n\right) \neq \emptyset$. The space X is called Choquet if Player 2 has a winning strategy.

Theorem. Complete metric spaces are Choquet.

Proof. Let Player 2 play open sets V_n s.t.

(i) $\overline{V_n} \subseteq U_n$

(ii) $\operatorname{diam}(V_n) \leq 2^{-n}$.

Then $\bigcap_n U_n = \bigcap_n \overline{V}_n \neq \emptyset$ by the completeness of X. □

Theorem. Choquet topological spaces are Baire.

Proof. By (3) of the equivalences to being Baire, it is enough to prove that a ctbl intersection $\bigcap_{n\geq 1} W_n$ of dense open sets W_n is dense. To this end, we fix a nonempty open set W_0 and aim to show that $W_0 \cap \bigcap_{n\geq 1} W_n = \bigcap_{n\geq 0} W_n \neq \emptyset$. Consider the following run of the Choquet game:

Player 1:	W_0	$W_1 \cap V_0$	$W_2 \cap V_1$	...
Player 2:	V_0	V_1	V_2	

where Player 2 plays according to her winning strategy, so $W_0 \cap \bigcap_{n\geq 1} (W_n \cap V_{n-1}) \neq \emptyset$, and hence $\bigcap_{n\geq 0} W_n \neq \emptyset$. □

Cor. In complete metric spaces (more generally, in Baire spaces), dense meagre sets are not G_δ. In particular, $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$ (it is a F_σ subset by def).

Cor. There is no function $f: \mathbb{R} \to \mathbb{R}$ that's continuous at every rational but discontinuous at every irrational.

Proof. The set of continuity pts of f is G_δ, hence cannot be $\mathbb{Q}$. $\square$